On a simple approach to nonlinear oscillators

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Abstract

We analyze a simple textbook approach to nonlinear oscillators proposed recently, disclose its errors, limitations and misconceptions and complete the calculations that the authors failed to perform.

1 Introduction

In a recent article Ren and He [1] proposed a simple method for the approximate calculation of the period of nonlinear oscillators and applied it to three rather trivial toy models. In this paper we discuss this remarkable scientific contribution that is another illustrative example of the new trend in mathematical physics promoted by certain journals.

2 Their method and our improvements

Ren and He [1] chose the dimensionless equation of motion

$$u'' + f(u) = 0 \tag{1}$$

with the initial conditions

$$u(0) = A,\, u'(0) = 0 \tag{2}$$

where the prime indicates differentiation with respect to the independent variable t.

The authors gave the following recipe: “We always choose cosine or sine function as a trial function for nonlinear oscillators. For the above initial conditions, we choose

$$u = A\cos\omega t \tag{3}$$

where ω is the angular frequency of the nonlinear oscillator to be further determined. Substituting Eq. (3) into Eq. (1) results in”

$$u'' + f(A\cos\omega t) = 0 \tag{4}$$

It is unbelievable that the authors did not realize that this equation does not apply to an arbitrary nonlinear oscillator for all values of t. In fact it is suitable only for the Hooke's force $f(u) = \omega^2 u$. The authors proceeded as follows: “Integrating Eq. (4) twice with respect to t, we have

$$u' = -\int_0^t f(u)\, dt \tag{5}$$

and”

$$u(t) = \int_0^t u'\, dt \tag{6}$$

Another incredible mistake, this equation does not satisfy the first initial condition in Eq (2).

In the discussion and conclusions Ren and He [1] admirably argued that "For an oscillator with initial conditions (2) we have"

$$u(0) = A,\ u(T/4) = 0,\ u(T/2) = -A,\ u(3T/4) = 0,\ u(T) = A \tag{7}$$

However, they did not bother to make it explicit that this set of equations is valid only for a nonlinear oscillator with odd force $f(-u) = -f(u)$. It is also probable that they were not aware of this obvious fact. Fortunately the authors only treated such particular cases.

Before proceeding with the discussion of this admirable piece of work, we recall a well known result in classical dynamics. If we multiply Eq. (1) by u' and integrate we obtain the textbook expression

$$\frac{u'^2}{2} + V(u) = E \tag{8}$$

where E is a constant of the motion and $dV/du = f$.

To continue with the discussion of the paper by Ren and He [1] we derive the correct expression that in our opinion is the basis of their approach. If we integrate Eq. (1) twice and take into account the boundary conditions already indicated above we obtain

$$u(t) = A - \int_0^t \int_0^{t'} f(u(t''))\, dt''\, dt' \tag{9}$$

that those authors supposedly tried to derive. The main idea behind their approach is that if one introduces an approximate solution into the right–hand side of Eq. (9) the result is expected to be an improvement. To obtain an approximate analytical expression for the period T the authors resorted to

the condition $u(T/4) = 0$ and in the conclusions they stated that "The suggested solution procedure is valid for conservation systems with unchanged amplitude". In fact, any equation of the form (1) represents a conservative system and exhibits a constant amplitude given by $V(u) = E$ that determines the turning points. Did the authors know it?. As indicated above $u(T/4) = 0$ is valid only for symmetric problems $V(-u) = V(u)$ or odd functions $f(-u) = -f(u)$. In fact, they only considered examples with that property without stating it explicitly in the general presentation of the approach. To facilitate the discussion below it is worth noticing that for an odd force $A \geq u(t) \geq 0$ if $0 \leq t \leq T/4$.

3 Their examples and our improvements

The first example chosen by Ren and He [1] is the well known and widely studied Duffing oscillator:

$$f(u) = u + \epsilon u^3 \tag{10}$$

Undergraduate students learn how to obtain satisfactory approximate solutions to this equation in most textbooks on classical mechanics. As shown below, the results of Ren and He [1] are of such kind, except that they were published in a research journal and were derived in a sloppy way. Apparently, those authors did not realize that the model parameters ϵ and A do not appear separately in the results but in the form of the only relevant quantity $\rho = \epsilon A^2$, an observation that greatly facilitates the discussion of the results. They first substituted Eq. (3) with $\omega = 2\pi/T$ into the right–hand side of Eq. (9) and obtained an improved trajectory as well as the approximate

period [1]

$$T^{[1]}(\rho) = \frac{2\pi}{\sqrt{1+\frac{7}{9}\rho}} \tag{11}$$

This expression provides a reasonable approach to the actual period for all values of ρ, and in particular an acceptable estimate of the limit

$$\lim_{\rho\to\infty} \sqrt{\rho}T(\rho) = T_\infty = 7.416298709 \tag{12}$$

In fact, Eq. (11) gives us $T_\infty^{[1]} \approx 7.12$.

How did they derive a reasonable result from the wrong equation (6)? Simply by the addition of another error that corrects the first one. Notice that the left– and right–hand sides of equations (10) and (11) in their paper do not match. They added convenient integration constants in the last steps to satisfy the boundary conditions.

In order to improve this first estimate Ren and He [1] then tried the ansatz

$$u(t) = A_1 \cos(\omega t) + A_2 \cos(3\omega t) \tag{13}$$

where $A_1 + A_2 = A$. The additional coefficient requires an additional condition and the authors chose

$$u''(0) + f(A) = 0 \tag{14}$$

In this way they derived a complicated system of equations that suspiciously they did not try any further. Besides, at first glance their results do not appear to be functions of ρ alone. For this reason in what follows we derive a suitable expression and verify if it is more accurate than the first approximation discussed above. Starting from the same premises our expression for

the period results to be

$$125T^8\rho\left(\rho+1\right)^3-7656\pi^2T^6\rho\left(\rho+1\right)^2+120\pi^4T^4\left(\rho+1\right)\left(1607\rho+735\right)$$
$$-64\pi^6T^2\left(48851\rho+55125\right)+12700800\pi^8=0 \quad (15)$$

This equation has four real roots $-T^{[2,2]}(\rho) < -T^{[2,1]}(\rho) < 0 < T^{[2,1]}(\rho) < T^{[2,2]}$, and $T^{[2,1]}(\rho)$ is the desired approximation that satisfies $T^{[2,1]}(0) = 2\pi$.

A straightforward calculation shows that the approximation to T_∞ is a root of

$$125T_\infty^8-7656\pi^2T_\infty^6+192840\pi^4T_\infty^4-3126464\pi^6T_\infty^2+12700800\pi^8=0 \quad (16)$$

We thus obtain $T_\infty^{[2,1]} \approx 7.44$ that is in fact more accurate than $T_\infty^{[1]}$.

If the Duffing equation is a textbook exercise, the second example studied by Ren and He [1], given by

$$f(u)=\epsilon\ sgn(u) \quad (17)$$

is ridiculously trivial because it is almost impossible to miss the exact result (remember that this paper is published in a research journal). We know that $sgn(u) = 1$ for $0 \leq t \leq T/4$; therefore, if we substitute $f(u) = \epsilon$ into the right hand–side of Eq. (9) we obtain $u(t) = A - \epsilon t^2/2$ and the exact period $T = 4\sqrt{2A/\epsilon}$. Any trial function that is positive definite in this interval leads to the same result, even if it does not satisfy the initial condition.

The third example is

$$f(u)=\omega_0^2u+\epsilon u|u| \quad (18)$$

but it is sufficient to consider $f(u) = \omega_0^2 u + \epsilon u^2$ by virtue of the argument given above. Notice that in this case the independent model parameters are ω_0 and $\rho = \epsilon A$. If we choose the trial function (3) we obtain their result

$$T^{[1]} = \frac{2\pi}{\sqrt{\omega_0^2 + \frac{4+\pi^2}{16}\rho}} \tag{19}$$

from which it follows that $T_\infty^{[1]} \approx 6.75$.

If we use the trial function (13) we easily obtain and improved expression for the period

$$\begin{aligned} & T^6 \rho \left(\omega_0^4 + 2\omega_0^2 \rho + \rho^2\right)\left(9\pi^2 - 16\right) \\ & -8\pi^2 T^4 \left[256\omega_0^4 + 15\omega_0^2 \rho \left(3\pi^2 + 16\right) + \rho^2 \left(45\pi^2 - 16\right)\right] \\ & +16\pi^4 T^2 \left[5120\omega_0^2 + \rho\left(369\pi^2 + 1136\right)\right] - 294912\pi^6 = 0 \end{aligned} \tag{20}$$

and the corresponding limit $\rho \to \infty$

$$T_\infty^6 \left(9\pi^2 - 16\right) + 8\pi^2 T_\infty^4 \left(16 - 45\pi^2\right) + 16\pi^4 T_\infty^2 \left(369\pi^2 + 1136\right) - 294912\pi^6 = 0 \tag{21}$$

In this case there are only two real roots $-T_\infty^{[2]}$ and $T_\infty^{[2]} \approx 6.867$ and the agreement of the positive one with the exact result $T_\infty = 6.868663935$ is remarkable.

4 Conclusions

The method proposed by Ren and He [1] could be thought of as an undergraduate exercise on classical mechanics with the limitation that it is valid only for odd forces because of the necessary condition $u(T/4) = 0$. Their paper

may be of some pedagogical value as an exercise for undergraduate students who may be asked to find as many mistakes as possible. The lecturer may even organize a kind of competition among groups of students dedicated to such a task. We think that it can be a hilarious class. The second example in that paper is trivial and, therefore, only useful as a curiosity for beginners.

We have carried out the second–order approximation for the Duffing oscillator that those authors left unfinished, and also showed how to do that calculation for the third example that they never tried. There is no need to say that the results of that paper have no serious utility whatsoever for actual research in the field of nonlinear oscillations. However, it is not surprising that such a sloppy paper had been published in a research journal where one finds many such examples. We have in fact discussed several of them in a series of communications [2–10]. In particular we want to draw the reader's attention to the extraordinary case of a predator–prey model that predicts a negative number of rabbits [5].

Finally, we mention that when a mild version of this article was submitted to the journal we were told " We urge you to contact the authors of this article before you submit a comment for publication. This approach is probably going to be much more productive for all concerned. I am going to reject the manuscript for now, but if after speaking directly with the authors you've come to an agreement that this manuscript should be published then you can resubmit." If we understand it clearly we are urged to ask permission from the authors to criticize their paper. Who would agree to it?

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